ARS 42-80
November 1963

SOME THEORETICAL ASPECTS OF SOIL FUMIGANT DIFFUSION

Agricultural Research Service
UNITED STATES DEPARTMENT OF AGRICULTURE

In agriculture, pesticides in both liquid droplet and solid particle form are used extensively, but their use is frequently hampered by lack of optimum efficiency and precision of application, despite the best efforts of manufacturers and applicators. This results in an excessive economic burden on agriculture in the maintenance of the best of quality in agricultural produce, as well as an increased change of contaminating surrounding areas. Problems of air pollution are also of prime pertinence and concern. It is desired that the results of these investigations in fine particle behavior and its inherent subject matter will serve agriculture and its allied industries, and other industries with similar vexations, in the alleviation of these problems.

SOME THEORETICAL ASPECTS OF SOIL FUMIGANT DIFFUSION

Ross D. Brazee and Orve K. Hedden [1]

Calculation of soil fumigant distribution patterns on the basis of diffusion theory is considered for several typical application configurations. The diffusion formalism of Hemwall is used, and techniques for solution are discussed. Instantaneous plane sources for one dimension and instantaneous line sources in two dimensions are essential to the formal approach.

INTRODUCTION

A number of workers have studied soil fumigation from the standpoint of diffusion theory. For example, Thorne (*8*),[2] Siegel, Erickson, and Turk (*7*), and Lange and Carlson (*5*) have made experimental studies of fumigant diffusion patterns. However, the papers of Goring (*2*), Pieczarka and Warren (*6*) and Hemwall (*3*, *4*) are of particular interest to this discussion. Goring outlined some of the pertinent diffusion concepts and discussed some experimental studies. The experimental work of Pieczarka and Warren dealt with the effect of surface sealants on fumigant diffusion. Hemwall developed a diffusion equation with a special diffusion coefficient formulation to account for the effects of soil porosity, temperature, organic matter, water, and mineral content on the fumigation process. Hemwall then calculated dosage patterns for a plane normal to instantaneous infinite line sources, and studied the effects of the various soil factors through their influences upon the diffusion coefficient. To surmount the difficulties of biological dosage computations, Hemwall used digital computer techniques.

Diffusion theory calculations can be expected to be useful in determining where the fumigant should be initially placed in order to treat a specified soil region. Hence, the purpose of the subsequent discussion will be to study some mathematical procedures for treatment of some typical fumigant application configurations on the basis of Hemwall's formalism.

This paper will consider the soil layers as homogeneous and isotropic with a fixed diffusion coefficient. The case of non-homogeneous soils or layered soils, i.e., composite media, though recognized, will not be treated at this time. Extensive use will be made of the unit instantaneous source, either as an infinite plane normal to the flow direction in the one-dimensional case, or as an infinite line perpendicular to the flow plane in two dimensions. These concepts are illustrated in figures 1, 2, and 3. Figure 1 shows the initial configuration, or condition, at $t=0$ later described (*see* Sec. 7) as a unit instantaneous plane source at depth x' in the region $0 < x < a$, with the soil surface sealed and an impermeable surface at $x=a$. A second variation is source and soil surface alone *with no* surface seal (surface loss of fumigant is ruled to occur whenever the seal is absent) and *without* the impermeable surface at $x=a$ so that essentially a semi-infinite medium exists. A third variant of this configuration would have the source-soil surface combination with surface seal *absent* but an impermeable barrier at $x=a$. The fourth has the source-soil surface combination with surface seal *present* and *without* an impermeable barrier below at $x=a$. Each of the configurations shown in figures 2 and 3 may be applied in four similar variations. The instantaneous "slab source" of figure 2 amounts to a combination of instantaneous plane sources. The instantaneous infinite line source is illustrated in figure 3.

The instantaneous source, as a mathematical model, is intended to represent the usual physical mode of fumigant application. In practice, a specified quantity of fumigant is injected at the desired point at $t=0$, and allowed to spread. This is precisely analogous to the nature of the mathematical instantaneous source. It is only necessary to multiply the unit source solution by the fumigant concentration per unit length, area, or volume as the case may be.

---

[1] Pioneering Research Laboratory on Physics of Fine Particles and Pest Control Equipment and Practices Section, respectively, Agricultural Engineering Research Division, ARS, USDA, Wooster, Ohio.

[2] Parenthetical numbers in italics refer to Literature Cited at end of publication.

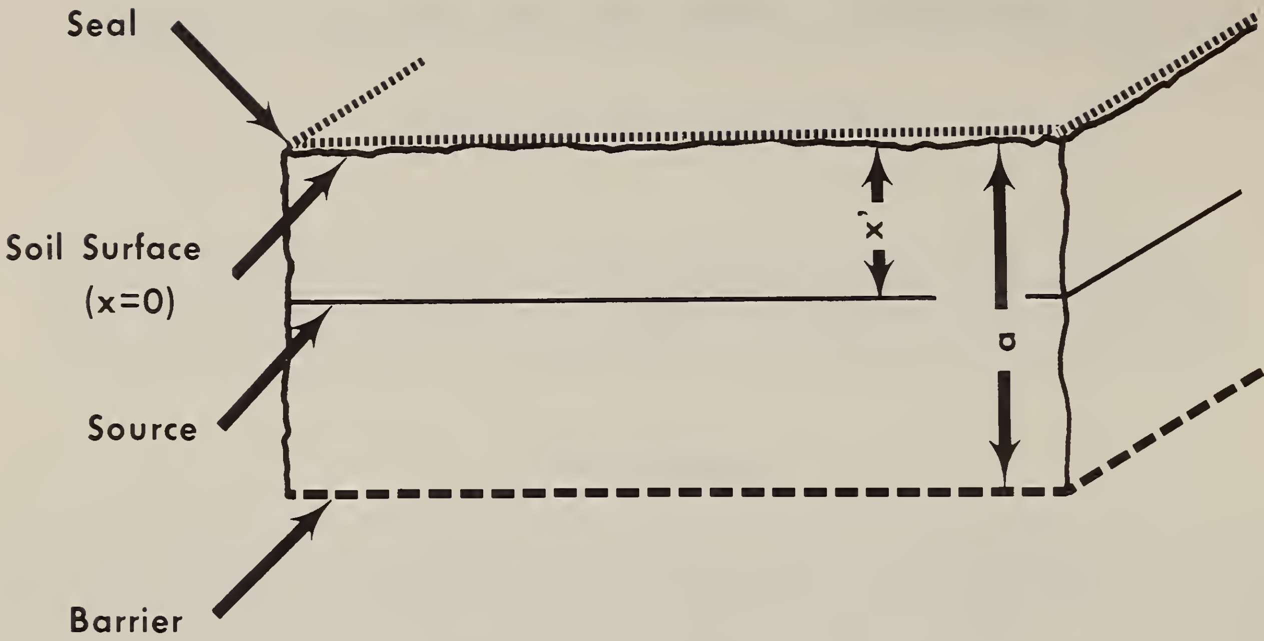

Figure 1.—Configuration concept for an instantaneous plane source at depth $x = x'$ from soil surface, with an impermeable barrier at depth $x = a$. Although an uneven soil surface is shown in the drawings, this factor is neglected mathematically; and in the case of sealants, intimate seal-soil contact is assumed.

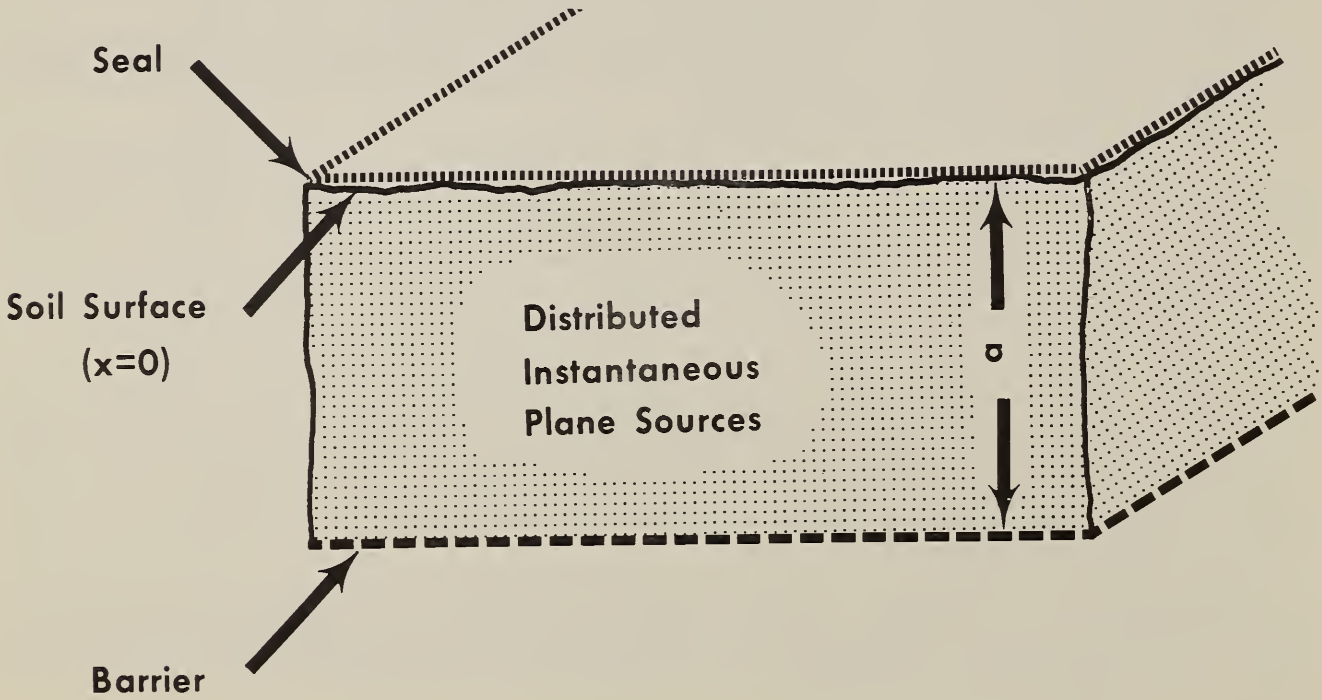

Figure 2.—Configuration for multilocated instantaneous plane sources distributed at $x' > 0$, $0 < x' < a$, with an impermeable barrier at depth $x = a$.

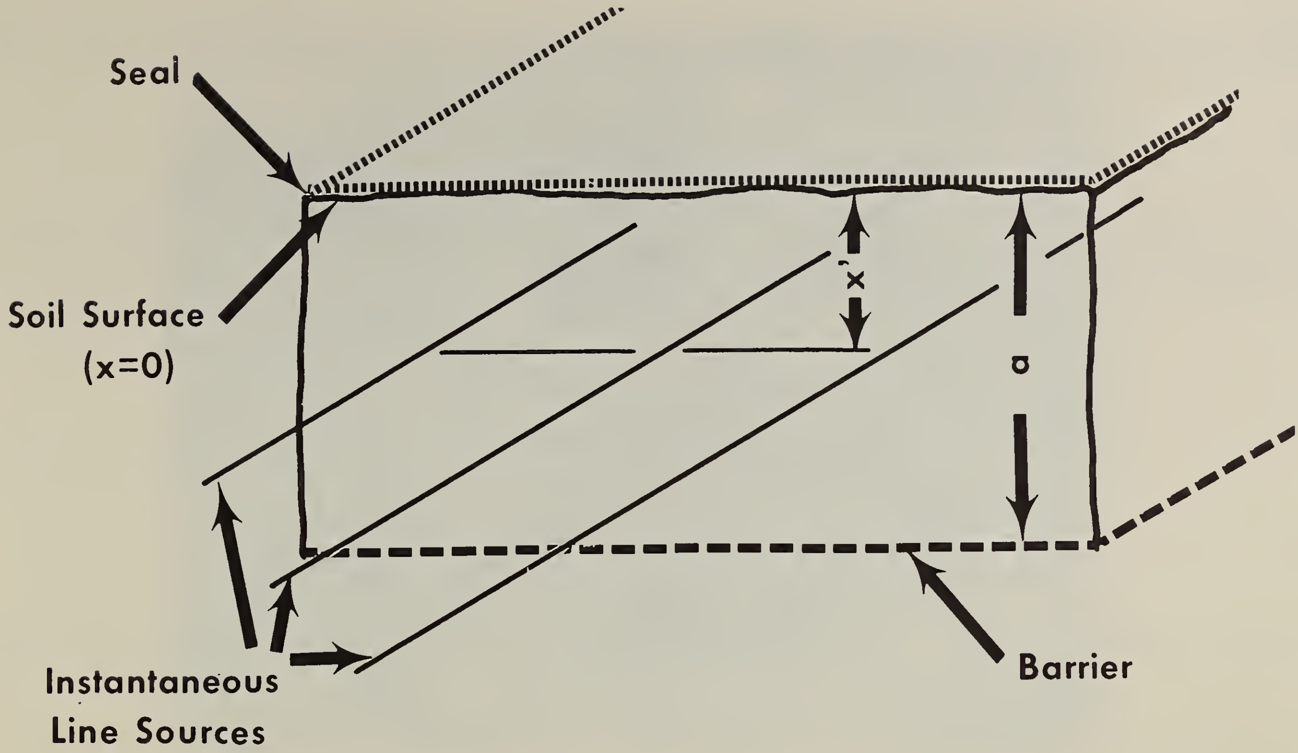

Figure 3.–Configuration for an array of instantaneous infinite line sources at depth $x = x'$, with an impermeable barrier at depth $x = a$.

An example shown in figure 4 corresponds to the condition discussed (*see* Sec. 9. I) in which the configuration of figure 2 is varied by utilizing only the source (rototilled into the soil), and a seal (polyethylene film) placed over the soil surface. In figure 5, the method of injection at a point behind a traveling blade (similar to the "field-cultivator" type of application) corresponds to the variation of the figure 3 configuration (*see* Sec. 12) for the unsealed soil surface-line source combination in the absence of the barrier at $x = a$.

Figure 4.—Fumigation treatment being rototilled into the soil and covered with a polyethylene surface seal.

Figure 5.—Application of a soil treatment in single lines in the root zone along each side of a row of growing plants.

The subsequent discussion deals basically with the solution of some boundary value problems, certain ones of which are standard in, e.g., heat conduction theory. The surface and barrier structures will set the boundary conditions, and the application configuration, or instantaneous source arrangement, will establish the initial conditions. The precise mathematical statement of the initial and boundary conditions will be covered in a later section (*see* Sec. 2).

It is important to be aware of certain advantages, and limitations, of the diffusion theory formulation of the fumigation process. The diffusion equation arises naturally from random flight considerations which could be expected to be appropriate to the process involved. As such, the equation represents the behavior of some average property of a many-particle ensemble and, has a very definite directionality in time. From theorems of thermodynamics, irregularities in such average properties, e.g., initial instantaneous sources, will be smoothed with the progression of time. The diffusion equation will thereby represent processes where entropy increases as time passes. In applying ordinary diffusion theory, one must recognize that in a specific problem, as time departs from $t=0$, the concentration of the diffusing substance will immediately take on some finite, though perhaps very small value at all points in the space of concern. This is, of course, physically impossible, and arises from the fact that the ordinary diffusion equation assumes an infinite velocity of propagation. However, this will be no serious difficulty in practical problems, particularly after a very short time period.

1. THE DIFFUSION EQUATION

Hemwall (*4*) uses a diffusion equation of the form

$$\frac{\partial c(x,y,t)}{\partial t}=D\left[\frac{\partial^2 c(x,y,t)}{\partial x^2}+\frac{\partial^2 c(x,y,t)}{\partial y^2}\right]-k_r c(x,y,t), \tag{1}$$

where $c(x, y, t)$ is the concentration of fumigant in the soil water, k_r is the rate constant for irreversible chemical decomposition of the fumigant in the soil, and the unsteady-state diffusion coefficient D is given by

$$D=\frac{0.66k_aD_0}{k_a+k_wK_hK_a}. \tag{2}$$

In (2), k_a is the volume fraction of the soil that is continuous air space; D_0 is the diffusion coefficient of the fumigant through air; k_w is the volume fraction of the soil that is water; K_h is the Henry's law constant defined as the ratio of the concentration of the fumigant in the soil water to the concentration in the soil air; and K_a is the adsorption constant defined as the ratio of the moles of fumigant adsorbed at the various solid surfaces per unit volume of soil to the concentration of the fumigant in the air space.

If the transformation

$$c(x, y, t)=u(x, y, t)e^{-k_rt}, \tag{3}$$

is defined, then (1) takes the standard form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{D}\frac{\partial u}{\partial t}, \tag{4}$$

which will be used throughout this paper. The same transformation will apply to the boundary conditions which are subsequently discussed.

2. TYPES OF BOUNDARY CONDITIONS AND SOLUTIONS

The initial and boundary conditions for the problems of concern will always be of the general form

$$c=\prod_i^2 f_i(x_i), t=0, a_i<x_i<b_i; \tag{5}$$

$$\alpha_i\frac{\partial c}{\partial x_i}-\beta_i c=0, x_i=a_i, t>0, i=1, 2; \tag{6}$$

$$\alpha_i'\frac{\partial c}{\partial x_i}+\beta_i' c=0, x_i=b_i, t>0, i=1, 2, \tag{7}$$

where the α_i, α_i', β_i and β_i' are constants. Hence, the discussion of Carslaw and Jaeger (*1, pp. 33–35*) on product solutions of the diffusion equation may be expected to apply, although not without care. By virtue of the transformation (3), the initial and boundary conditions will generally appear in the form

$$u=\prod_i^2 f_i(x_i), t=0, a_i<x_i<b_i; \tag{8}$$

$$\alpha_i\frac{\partial u}{\partial x_i}-\beta_i u=0, x_i=a_i, t>0, i=1, 2; \tag{9}$$

$$\alpha_i'\frac{\partial u}{\partial x_i}+\beta_i' u=0, x_i=b_i, t>0, i=1, 2, \tag{10}$$

In this paper, it will be assumed that the rate of fumigant loss through the soil surface into the atmosphere (in which the fumigant concentration effectively vanishes) is linearly proportional to the concentration just inside the surface. The mathematical statement of this condition is

$$\frac{\partial u}{\partial x}=hu, x=0, t>0, \tag{11}$$

where $x=0$ is taken at the soil surface and h is a constant. This is analogous to the linear "radiation" boundary condition of the theory of heat conduction (*1, pp. 18–19*). If a plastic film or other sealant

which may be considered impenetrable is placed on the soil surface, the boundary condition statement becomes

$$\frac{\partial u}{\partial x}=0,\ x=0,\ t>0. \tag{12}$$

The vertical downward direction will usually be taken along the positive x-axis of coordinates. If the soil is pervious to the fumigant over a considerable depth, the region of interest will be $x>0$. However, if the upper regions of the soil are underlaid by a relatively impervious layer, then the region of interest will be $0<x<a$, and for the present discussion we will presume to write

$$\frac{\partial u}{\partial x}=0,\ x=a,\ t>0, \tag{13}$$

with the barrier at depth a.

3. THE UNIT INSTANTANEOUS SOURCE [3]

In one-dimension, the unit instantaneous plane source at x' normal to the direction of flow is written

$$\frac{1}{2(\pi Dt)^{\frac{1}{2}}}e^{-(x-x')^2/4Dt}. \tag{14}$$

In two-dimensional problems, the unit instantaneous line source at $(x',\ y')$ normal to the plane of flow is written

$$\frac{1}{4\pi Dt}e^{-[(x-x')^2+(y-y')^2]/4Dt}. \tag{15}$$

4. GREEN'S FUNCTIONS IN THE SOLUTION OF THE EQUATION OF DIFFUSION

In this section, the Green's function method for solving the diffusion equation will be outlined. The terminology of Carslaw and Jaeger (1, *p. 356*) will be followed.

Two basic types of Green's function are appropriate to consider in diffusion theory. First, considering the two-dimensional case, we may take the Green's function, $G(x, y, t|x', y', \tau)$, to be the concentration of diffusing substance at $(x,\ y)$ at time t due to a unit instantaneous line source generated at $(x',\ y')$ at time τ, with the concentration initially at zero throughout the region and the bounding curve held at zero concentration. Secondly, the Green's function may be taken as the concentration at $(x,\ y)$ at time t due to a unit instantaneous line source generated at $(x',\ y')$ at time τ, with the boundary normal gradient of concentration specified in the manner $\frac{\partial u}{\partial n_i}=hu$. In either case, it is found that the Green's function solution of (4) for a region bounded by a curve s is

$$u(x, y, t)=\int\int[G(x, y, t|x', y', \tau)]_{\tau=0}f(x', y')\,dx'dy'+D\int_0^t\left[\int\phi(x', y', \tau)\frac{\partial u}{\partial n_i}ds'\right]d\tau, \tag{16}$$

where $f(x,\ y)$ is the initial distribution of diffusing material, $\phi(x,\ y,\ t)$ is the surface concentration, and $\partial/\partial n_i$ signifies differentiation along the inward normal to s. For the one-dimensional case, statement (16) becomes

$$u(x, t)=\int[G(x, t|x', \tau)]_{\tau=0}f(x')dx'+D\int_0^t\phi(\tau)\frac{\partial u}{\partial n_i}d\tau. \tag{17}$$

[3]See pp. 255–259 of Literature Cited 1.

The physical interpretations of (16) and (17) with respect to soil fumigant diffusion are simple but important: The concentration at time t in the soil region with initial concentration $f(x, y)$ (or $f(x)$) and zero concentration just outside the surface is obtained from a distribution of instantaneous sources at $t=0$ over the region, with an amount of substance $f(x, y)\, dx\, dy$ (or $f(x)\, dx$) being liberated in the element $dx\, dy$ (or dx) at x, y (or x). Hence, when the Green's function for the given region and boundary conditions is known, the solution of the problem for an initial concentration which is an arbitrary function of position can be immediately written on the basis of the forms (16) and (17).

The Green's function method will be found helpful in several of the problems of this paper. In some cases, the problems consist largely of finding the Green's function. We now proceed to the treatment of specific problems, starting with one-dimensional flow.

5. UNIT INSTANTANEOUS PLANE SOURCE AT DEPTH x' AT $t=0$ IN THE REGION $x>0$. SOIL SURFACE SEALED

The appropriate form of the diffusion equation is

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{D}\frac{\partial u}{\partial t}, \tag{18}$$

with the boundary condition

$$\frac{\partial u}{\partial x}=0,\ x=0,\ t>0. \tag{19}$$

The solution may be written down immediately as

$$u(x, t)=\frac{1}{2(\pi Dt)^{1/2}}\left[e^{-(x-x')^2/4Dt}+e^{-(x+x')^2/4Dt}\right] \tag{20}$$

by the method of images (*1, p. 273*). By transformation (3), the solution in terms of fumigant concentration is

$$c(x, t)=\frac{e^{-k_r t}}{2(\pi Dt)^{1/2}}\left[e^{-(x-x')^2/4Dt}+e^{-(x+x')^2/4Dt}\right]. \tag{21}$$

6. UNIT INSTANTANEOUS PLANE SOURCE AT DEPTH x' AT $t=0$ IN THE REGION $x>0$. SOIL SURFACE UNSEALED

The boundary condition is

$$\frac{\partial u}{\partial x}=hu,\ x=0,\ t>0. \tag{22}$$

Carslaw and Jaeger (*1, 358–359*) outline the Laplace transformation solution in detail. The result is

$$u(x, t)=\frac{1}{2(\pi Dt)^{1/2}}\left[e^{-(x-x')^2/4Dt}+e^{-(x+x')^2/4Dt}\right]-he^{Dth^2+h(x+x')}\operatorname{erfc}\left[\frac{x+x'}{2(Dt)^{1/2}}+h(Dt)^{1/2}\right], \tag{23}$$

which for fumigant concentration becomes

$$c(x, t)=e^{-k_r t}\left\{\frac{1}{2(\pi Dt)^{1/2}}\left[e^{-(x-x')^2/4Dt}+e^{-(x+x')^2/4Dt}\right]-he^{Dth^2+h(x+x')}\operatorname{erfc}\left[\frac{x+x'}{2(Dt)^{1/2}}+h(Dt)^{1/2}\right]\right\}. \tag{24}$$

In (23) and (24), the expression

$$erfc\left[\frac{x+x'}{2(Dt)^{1/2}}+h(Dt)^{1/2}\right]=\frac{2}{(\pi)^{1/2}}\int_{\frac{x+x'}{2\sqrt{Dt}}+h\sqrt{Dt}}^{\infty} e^{-\xi^2}d\xi \tag{25}$$

is the error function complement.

7. UNIT INSTANTANEOUS PLANE SOURCE AT DEPTH x' AT $t=0$ IN THE REGION $0<x<a$. SOIL SURFACE SEALED AND AN IMPERMEABLE SURFACE AT $x = a$

The boundary conditions are

$$\frac{\partial u}{\partial x}=0,\ x=0 \text{ and } x=a,\ t>0. \tag{26}$$

We first write

$$u=v+w, \tag{27}$$

where

$$v=\frac{1}{2(\pi Dt)^{1/2}}e^{-(x-x')^2/4Dt}, \tag{28}$$

and w is a solution of the one-dimensional diffusion equation which approaches zero as t approaches zero and is chosen so that the boundary conditions (26) on u are satisfied.

The Laplace transform of (27) is

$$\mathcal{L}\{u\}=U=V+W, \tag{29}$$

and for (28) is

$$V=\frac{1}{2qD}e^{-q|x-x'|}, \tag{30}$$

where

$$q=(p/D)^{1/2}. \tag{31}$$

The subsidiary equation for W is

$$\frac{d^2W}{dx^2}-q^2W=O.$$

Hence, U is given by

$$U=\frac{1}{2qD}e^{-q|x-x'|}+A\sinh qx+B\cosh qx, \tag{32}$$

where A and B are arbitrary constants. Upon applying the boundary conditions (26) and simplifying the results, we find that U takes the form

$$U=\frac{\cosh q(a+x-x')+\cosh q(a-x-x')}{2Dq\sinh qa},\ 0<x<x', \tag{33}$$

and

$$U=\frac{\cosh q(a+x'-x)+\cosh q(a-x'-x)}{2Dq\sinh qa},\ x'<x<a. \tag{34}$$

The inversion theorem may be applied to determine $u(x, t)$, viz.,

$$u(x,t)=\mathcal{L}^{-1}\{U(x,p)\}=\lim_{R\to\infty}\left\{\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR}\frac{e^{pt}[\cosh q(a+x-x')+\cosh q(a-x-x')]dp}{2Dq\sinh qa}\right\}, \tag{35}$$

where (33) is to be considered first. The integrand in (35) has simple poles at

$$p=0;\ p=-Dn^2\pi^2/a^2,\ n=1,2,\ .\ .\ . \tag{36}$$

We select a contour of the form shown in figure 6 and large enough to contain all the poles of the integrand. The residues at the poles of the integrand are $(1/a)$ for $p=0$, and

$$\operatorname{Res}\left\{\frac{e^{pt}[\cosh q(a+x-x')+\cosh q(a-x-x')]}{2Dq\sinh qa}\right\}_{q=in\pi/a}$$

$$=\left\{\frac{e^{-Dn^2\pi^2t/a^2}\left[\cosh\left(\dfrac{in(a+x-x')\pi}{a}\right)+\cosh\left(\dfrac{in(a-x-x')\pi}{a}\right)\right]}{2D[(d/dp)\ ((p/D)^{1/2}\sinh(a(p/D)^{1/2}))]}\right\}_{p=-Dn^2\pi^2/a^2}$$

$$=\frac{e^{-Dn^2\pi^2t/a^2}}{a\cos(n\pi)}\left[\cos\frac{(a+x-x')n\pi}{a}+\cos\frac{(a-x-x')n\pi}{a}\right]. \tag{37}$$

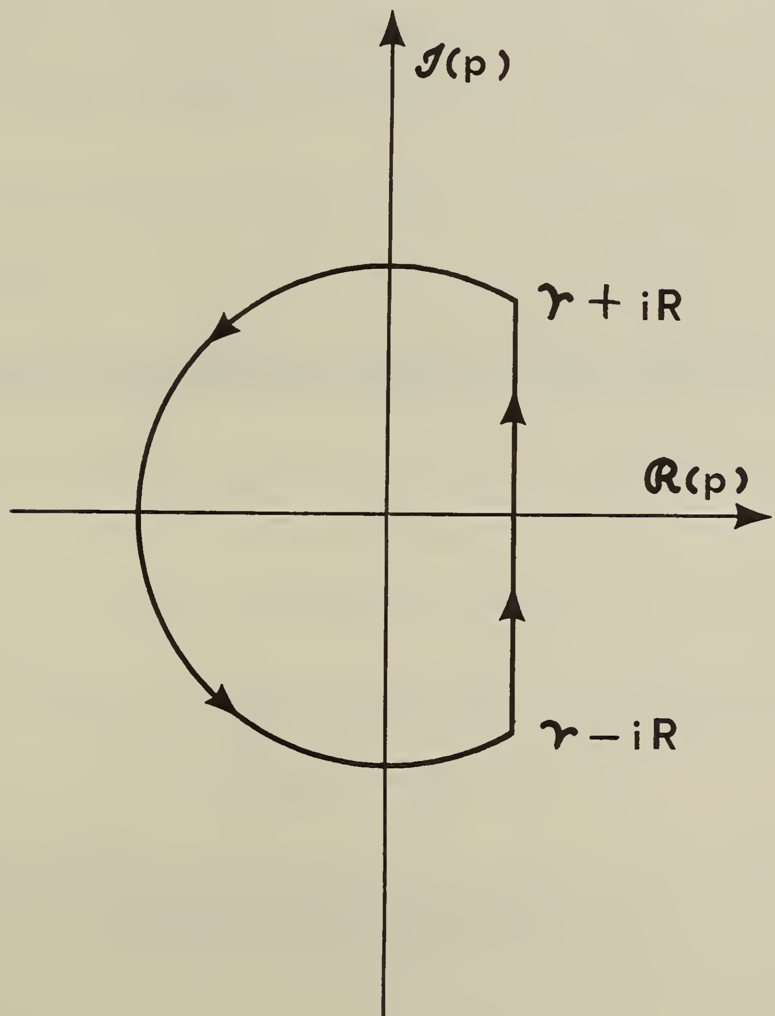

FIGURE 6

Then the sum of the residues inside the contour will be

$$\Sigma \text{ Res} = \frac{1}{a} + \frac{2}{a} \sum_{n=1}^{\infty} e^{-Dn^2\pi^2 t/a^2} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x'}{a}\right). \tag{38}$$

We have for the integral on the contour of figure 6

$$\frac{1}{2\pi i}\left\{\int_{\gamma+i\infty}^{\gamma-i\infty} \frac{e^{pt}[\cosh q(a+x-x') + \cosh q(a-x-x')]dp}{2Dq \sinh qa}\right\} = \sum \text{Res}, \tag{39}$$

since the integral along the large circle is zero. Hence, for the inverse transform we have

$$u(x, t) = \frac{1}{a}\left[1 + 2\sum_{n=1}^{\infty} e^{-Dn^2\pi^2 t/a^2} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x'}{a}\right)\right]. \tag{40}$$

An identical result is found for the transform (34), which indicates (40) is a complete solution in terms of $u(x, t)$.

The solution to our problem in the notation of fumigant concentration is, therefore,

$$c(x, t) = \frac{e^{-k_r t}}{a}\left[1 + 2\sum_{n=1}^{\infty} e^{-Dn^2\pi^2 t/a^2} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x'}{a}\right)\right]. \tag{41}$$

8. UNIT INSTANTANEOUS PLANE SOURCE AT DEPTH x' AT $t=0$ IN THE REGION $0 < x < a$. SOIL SURFACE UNSEALED AND IMPERMEABLE SURFACE AT $x=a$

The boundary conditions are

$$\frac{\partial u}{\partial x} = hu, \; x = 0, \; t > 0; \tag{42}$$

$$\frac{\partial u}{\partial x} = 0, \; x = a, \; t > 0. \tag{43}$$

The solution may be obtained by Laplace transform methods, but we shall merely adapt the general result of Carslaw and Jaeger (*1, p. 360*) and write

$$u(x, t) = \sum_{n=1}^{\infty} Z_n(x) Z_n(x') e^{-D\alpha_n^2 t}, \tag{44}$$

where

$$Z_n(x) = \frac{(2)^{1/2} \alpha_n[\alpha_n \cos(\alpha_n x) + h \sin(\alpha_n x)]}{\{(\alpha_n^2 + h^2)[a\alpha_n^2 + h\alpha_n]\}^{1/2}} \tag{45}$$

and $\pm\alpha_n$, $n=1, 2, \ldots$, are roots of

$$\alpha \tan (\alpha a) = h. \tag{46}$$

Carslaw and Jaeger (*1, p. 491*) give tables of the first six roots of the transcendental equation (46).
The solution in terms of fumigant concentration is

$$c(x, t) = e^{-k_r t} \sum_{n=1}^{\infty} Z_n(x) Z_n(x') e^{-D\alpha_n^2 t}. \tag{47}$$

9. SOLUTIONS FOR ARBITRARY INITIAL DISTRIBUTIONS OF CONCENTRATION

In obtaining the solutions for $u(x, t)$, in Sections 5 to 8, we have at the same time obtained Green's functions $G(x, t|x', \tau)$ for the regions of concern. Hence, it is possible to write down a series of solutions for arbitrary initial distributions of fumigant by using (17) and the appropriate Green's function. Mathematically, the initial condition will be

$$c(x, 0) = f(x), \; 0 < x < a \text{ or } \infty \tag{48}$$

The solutions for several regions follow.

I. *The region $x > 0$ with the soil surface sealed. Boundary condition of Section 5.*

$$c(x, t) = \frac{e^{-k_r t}}{2(\pi D t)^{1/2}} \int_0^{\infty} [e^{-(x-x')^2/4Dt} + e^{-(x+x')^2/4Dt}] f(x') dx'. \tag{49}$$

II. *The region $x > 0$ with the soil surface unsealed. Boundary condition of Section 6.*

$$c(x, t) = e^{-k_r t} \int_0^{\infty} \left\{ \frac{1}{2(\pi D t)^{1/2}} [e^{-(x-x')^2/4Dt} + e^{-(x+x')^2/4Dt}] - h e^{Dth^2 + h(x+x')} \operatorname{erfc} \left[\frac{x+x'}{2(Dt)^{1/2}} + h(Dt)^{1/2} \right] \right\} f(x') dx'. \tag{50}$$

III. *The region $0 < x < a$ with the surface sealed and an impermeable surface at $x = a$. Boundary conditions of Section 7.*

$$c(x, t) = \frac{e^{-k_r t}}{a} \int_0^a \left[1 + 2 \sum_{n=1}^{\infty} e^{-Dn^2\pi^2 t/a^2} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x'}{a}\right) \right] f(x') dx'. \tag{51}$$

IV. *The region $0 < x < a$ with the surface unsealed and an impermeable surface at $x = a$. Boundary conditions of Section 8.*

$$c(x, t) = e^{-k_r t} \sum_{n=1}^{\infty} e^{-D\alpha_n^2 t} \left\{ \int_0^a Z_n(x) Z_n(x') f(x') dx' \right\}. \tag{52}$$

The expressions for $Z_n(x)$ and α_n are as given in (45) and (46), respectively. The validity of the interchange of the operations of integration and summation has been assumed.

10. TWO-DIMENSIONAL CONFIGURATIONS

We consider a unit instantaneous line source at $x=x'$ and $y=y_j'$. This source may be written as in (15),

$$\frac{1}{4\pi Dt} e^{-[(x-x')^2+(y-y_j')^2]/4Dt}. \tag{53}$$

With the help of this general form and the Green's functions derived in the previous cases, we will attempt to write down some solutions for the two-dimensional line-source array. It is assumed in all cases that at great distances, the concentration approaches zero. Positive or negative integers are assigned to j as y_j' is positive or negative, and in the order

$$y_{-m}' < \ldots < y_{-2}' < y_{-1}' < 0 < y_1' < y_2' < \ldots y_n'. \tag{54}$$

11. UNIT INSTANTANEOUS SOURCE ARRAY AT DEPTH x' AT $t=0$ IN THE REGION $x>0$, $-\infty<y<\infty$. SOIL SURFACE SEALED

From the boundary condition

$$\frac{\partial u}{\partial x}=0, x=0, t>0, \tag{55}$$

and equations (20) and (53), we write down the solution

$$c(x,y,t)=\frac{e^{-k_r t}}{4\pi Dt}\sum_{j=-M}^{N}\left[e^{-(x-x')^2/4Dt}+e^{-(x+x')^2/4Dt}\right]e^{-(y-y_j')^2/4Dt}. \tag{56}$$

12. UNIT INSTANTANEOUS SOURCE ARRAY AT DEPTH x' AT $t=0$ IN THE REGION $x>0$, $-\infty<y<\infty$. SOIL SURFACE UNSEALED

By the boundary condition (22), and equations (23) and (53) we write

$$c(x,y,t)=\frac{e^{-k_r t}}{2(\pi Dt)^{1/2}}\sum_{j=-M}^{N}\left\{\frac{1}{2(\pi Dt)^{1/2}}\left[e^{-(x-x')^2/4Dt}+e^{-(x+x')^2/4Dt}\right]\right.$$

$$\left.-he^{Dth^2+h(x+x')}\operatorname{erfc}\left[\frac{x+x'}{2(Dt)^{1/2}}+h(Dt)^{1/2}\right]\right\}e^{-(y-y_j')^2/4Dt}. \tag{57}$$

13. UNIT INSTANTANEOUS SOURCE ARRAY AT DEPTH x' AT $t=0$ IN THE REGION $0<x<a$, $-\infty<y<\infty$. SOIL SURFACE SEALED AND AN IMPERMEABLE SURFACE AT $x=a$.

From the boundary conditions (26), and equations (40) and (53) we have

$$c(x,y,t)=\frac{e^{-k_r t}}{2a(\pi Dt)^{1/2}}\sum_{j=-M}^{N}\left\{1+2\sum_{n=1}^{\infty}e^{-Dn^2\pi^2 t/a^2}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi x'}{a}\right)\right\}e^{-(y-y_j')^2/4Dt}. \tag{58}$$

14. UNIT INSTANTANEOUS SOURCE ARRAY AT DEPTH x' AT $t=0$ IN THE REGION $0<x<a$, $-\infty<y<\infty$. SOIL SURFACE UNSEALED AND AN IMPERMEABLE SURFACE AT $x=a$

On the basis of the boundary conditions (42) and (43), and equations (44) and (53) we obtain

$$c(x, y, t)=\frac{e^{-k_r t}}{2(\pi D t)^{1/2}} \sum_{j=-M}^{N} \left\{ \sum_{n=1}^{\infty} Z_n(x) Z_n(x') e^{-D\alpha^2 t} \right\} e^{-(y-y_j')^2/4Dt}, \tag{59}$$

with $Z_n(x)$ and α_n^2 as expressed by (45) and (46), respectively.

LITERATURE CITED

(1) Carslaw, H. S., and Jaeger, J. C. Conduction of Heat in Solids. Ed. 2, 510 pp. London. 1959.

(2) Goring, C. A. I. Factors Influencing Diffusion and Nematode Control by Soil Fumigants. Dow Chemical Company Agricultural Chemical Development Information Bulletin No. 110, 53 pp. 1957.

(3) Hemwall, J. B. A Mathematical Theory of Soil Fumigation. Soil Sci. 88: 184–190. 1959.

(4) Hemwall, J. B. Theoretical Considerations of Several Factors Influencing the Effectivity of Soil Fumigants Under Field Conditions. Soil Sci. 90: 157–168. 1960.

(5) Lange, W. H., Jr., and Carlson, E. C. Zonal Dispersion of Chemicals in Soil Following Several Tillage Methods. J. Econ. Entom. 48: 61–67. 1955.

(6) Pieczarka, S. J., and Warren, G. F. Effect of Soil Seals on the Diffusion and Drench Patterns of Fumigants. Soil Sci. 91: 265–271. 1961.

(7) Siegel, J. J., Erickson, A. E., and Turk, L. M. Diffusion Characteristics of 1–3 Dichloropropene and 1–2 Dibromoethane in Soils. Soil Sci. 72: 333–340. 1951.

(8) Thorne, G. Diffusion Patterns of Soil Fumigants. Proc. Helminthological Soc. Washington 18: 18–24. 1951.

U.S. GOVERNMENT PRINTING OFFICE : 1963 OL—692-685